Transport in Porous Media **30:** 301–321, 1998.

Heat Transfer in Nonsaturated Porous Media: Modelling by Homogenisation

J.-F. BLOCH[1] and J.-L. AURIAULT[2]
[1]*LGP2/INPG/CNRS UMR 5518, Domaine Universitaire, BP 65, 38402 St Martin d'Hères, France*
[2] *3S/UJF/INPG/CNRS UMR 5521, Domaine Universitaire, BP 53 X, 38041 St Martin d'Hères, France*

(Received: 9 December 1996; in final form: 6 January 1998)

Abstract. This paper is devoted to the modelling of a temperature field in nonsaturated porous media in the absence of phase change. We establish the energy equation at the macroscopic level, from a description at the pore level by using the homogenisation method of multiple-scale asymptotic expansions. Different macroscopic models are obtained depending on the values of the local Péclet number and the local Fourier number. An example of the application of the different model catalogue is presented which concerns the modelling of the hot pressing of a paper web.

Key words: temperature, homogenisation, modelling, unsaturated.

Notations

C	$C = \rho . C_p (\mathrm{J\,m^{-3}\,K^{-1}})$
C_p	specific heat ($\mathrm{J\,kg^{-1}\,K^{-1}}$)
I	identity matrix
l, L	characteristic microscopic and macroscopic size, respectively
n	volume ratio
$\mathbf{N}$	unit normal vector
Pe, P, N_λ	dimensionless numbers
t	time (s)
T	temperature (K)
$\mathbf{v}$	velocity ($\mathrm{m\,s^{-1}}$)
$\mathbf{x}$, $\mathbf{y}$	dimensionless macro and micro length, respectively
$\mathbf{X}$	dimensional space variable
α	thermal diffusivity ($\mathrm{m^2\ s^{-1}}$)
ε	scale factor
Γ_{kl}	interface between the domains filled by the phases k and l
λ	thermal conductivity ($\mathrm{J\,m^{-1}\,s^{-1}\,K^{-1}}$)
Ω, Ω_i	total volume, volume filled by the phase i ($\mathrm{m^3}$)
ρ	density ($\mathrm{kg\ m^{-3}}$)

Subscripts w, a, s are, respectively, liquid (water), gas (air) and solid phase.

$$\langle * \rangle_{\Omega k} = \frac{1}{|\Omega k|} \int_{\Omega k} * \, \mathrm{d}\Omega$$

$$\langle \mathbf{V} \rangle = \frac{1}{|\Omega|} \int_{\Omega k} \mathbf{V} \, \mathrm{d}\Omega = n \langle \mathbf{V} \rangle_{\Omega k} \quad (k = \mathrm{w, a, s})$$

1. Introduction

Improving the mechanical pressing yields substantial profits in papermaking by reducing the high-cost drying stage. Many works are devoted to this subject (Campbell, 1947; Wahlstrom, 1960; Wrist, 1964; Nilsson and Larsson, 1968; Fekete, 1975; Carlsson *et al*., 1983; Roux, 1986; Kataja *et al*., 1992). Increasing the temperature of the wet web leads to a better dewatering due to the modification of both the Young modulus of the fibrous network and the water viscosity (Back, 1985). The modelling of the temperature field in a paper web during pressing is obviously of primary interest in optimising the process. The temperature level in the technology investigated here (hot pressing) is not sufficient enough to introduce an important phase change in the paper web as well as on the contact surface between the paper web and the hot rolls. Therefore, the phase change that may nevertheless appear, will be neglected as a first approximation. The improvement of such a technology is essentially based on the modification of experimental pilots. In order to bypass some expensive experiments, a numerical modelling of the press section is to be hoped. In this aim, our paper tries to steer the numerical resolution (by indicating the correct choice of the thermal problem equations to be solved) dedicated to the industrial problem.

More generally, we investigate the modelling of the temperature field in a non-saturated porous medium. Due to the complex structure at the pore scale, a useful modelling is only possible at a macroscopic scale, where an equivalent continuous medium is defined. The macroscopic modelling can then be solved in the boundary-value problem constituted by the paper web between two rolls during pressing. Solving this problem will enable us to optimise the process. This later extent is the subject of intensive research (Bloch, 1995), out of the scope of the present work.

To obtain the macroscopic modelling, we start from a description at the pore level and we use the method of multiple-scale expansions (Bensoussan *et al*., 1978). The method used here is different from the classical average ones referenced in Kaviany (1991). Compared to them the method of multiple scale expansions shows several advantages (Auriault, 1991a):

– It needs first a local description, and secondly, a scale separation. Macroscopic modellings are then deduced from the local description, only, without any prerequisites concerning the macroscopic description. The volume averaging process is not arbitrarily introduced in the process; it is a consequence of the scale separation.
– It permits us to demonstrate, from a given local description, the existence, or the nonexistence, of a macroscopic equivalent description, i.e., the homogenisability or the nonhomogenisability of an heterogeneous medium. It also gives the domain of validity of the macroscopic description.
– It permits us to investigate the physical meaning of the macroscopic physical quantities. For example, considering a Darcy flow through a porous medium, the process introduces the volume average of the local fluid velocity in the pores.

It is possible to show that the volume average velocity is a surface average velocity, that makes this quantity physically meaningful (Auriault, 1991b).

A catalogue of different modellings is obtained depending on the values of the dimensionless numbers introduced by the pore-scale description. The adapted modelling for an actual problem is then determined from the characteristic values of the parameters that are involved in the studied situation.

The paper is divided as follows. Section 2 is devoted to the presentation of the pore-scale description and to the dimensionless numbers of interest. Characteristic values for a paper web saturated by air and water are then given in Section 3. These permit us to determine some of these dimensionless numbers. We are left with two variable dimensionless numbers, the Péclet and Fourier numbers, that are functions of the scale ratio in the medium. Then in Section 4, the different macroscopic modellings are given corresponding to different values of these two dimensionless numbers. An example of the use of the result concerning an actual paper web pressing process is presented in Section 5.

Information about the homogenisation process is given in the Appendix for a particular case of interest. However, the reader interested in the details of the analysis will refer to Auriault (1991a) or to the papers already published in *Transport in Porous Media* as Auriault *et al.* (1992, 1995).

2. The Pore-Scale Description

The non-saturated porous medium under consideration is composed of the porous structure(s), the wetting phase (w = water) and the nonwetting phase (a = air). We will denote Ω_s, Ω_w, Ω_a the volumes occupied at the microscale by the solid, water, and air phases, respectively. We assume each volume as one connected. The distribution of the fluid constituents are supposed to be annular, i.e., there is no contact between air and solid, as shown in Figure 1.

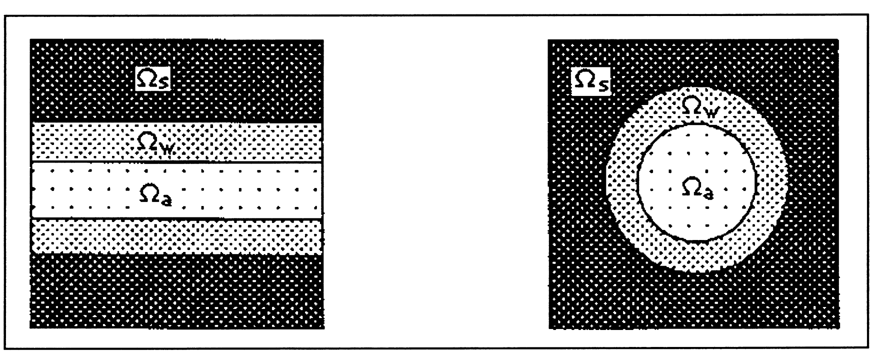

Figure 1. Schematic view of a pore.

The energy equation for a domain Ω_k $(k = \mathrm{w, a, s})$ is given by

$$\frac{\partial \rho_k C_{p_k} T_k}{\partial t} + \mathbf{v}_k \cdot \nabla(\rho_k C_{p_k} T_k) = \nabla \cdot (\lambda_k \nabla T_k), \tag{1}$$

where ρ, C_p, T, $\mathbf{v}$, t and λ are the density, the thermal capacity, the temperature, the velocity, the time, and the thermal conductivity, respectively.

In the absence of phase change, the flux conservation of heat on the interface Γ_{kl}, (between the domains k and l), whose unit normal is $\mathbf{N}$, is written in the form

$$\lambda_{k_{ij}} \left(\frac{\partial T_k}{\partial X_j} \right) N_i = \lambda_{l_{ij}} \left(\frac{\partial T_l}{\partial X_j} \right) N_i. \tag{2}$$

We also suppose that the thermal resistance is disregarded at the interfaces. We then deduce that the temperatures are continuous on each interface Γ_{kl}:

$$T_k = T_l. \tag{3}$$

The medium is assumed as periodic. Random media yield similar macroscopic description modellings (Auriault, 1991b). The method is based on the existence of two well-separated characteristic lengths: l is the characteristic pore size and L is, e.g., the macroscopic characteristic length of the sample, with $(l/L) = \varepsilon \ll 1$. The process may be summarised as follows:

(a) make the local description dimensionless.
(b) evaluate the dimensionless numbers in term of the scale separation parameter ε.
(c) look for the quantities in the form of asymptotic expansions with respect to the powers of ε.
(d) solve the successive boundary-value problems obtained by identifying like powers of ε.
(e) obtain the macroscopic description from the existence condition of the different terms in the asymptotic expansions.

To make Equations (1), (2) dimensionless, some parameters may be introduced (thermal conductivity ratios, inverse P of the Fourier number and Péclet number Pe):

$$\mathrm{N}_{\lambda_{\mathrm{aw}}} = \frac{\lambda_{\mathrm{a}}}{\lambda_{\mathrm{W}}}, \qquad \mathrm{N}_{\lambda_{\mathrm{sw}}} = \frac{\lambda_{\mathrm{s}}}{\lambda_{\mathrm{W}}},$$

$$\mathrm{P} = \frac{\left| \rho C_p \dfrac{\partial T}{\partial t} \right|}{\left| \dfrac{\partial}{\partial X_i} \lambda_{ij} \dfrac{\partial T}{\partial X_j} \right|}, \qquad \mathrm{Pe} = \frac{\left| \rho C_p V_i \dfrac{\partial T}{\partial X_i} \right|}{\left| \dfrac{\partial}{\partial X_i} \lambda_{ij} \dfrac{\partial T}{\partial X_j} \right|}.$$

The Péclet number links the mechanical and thermal aspects, and P, the inverse of the Fourier number, is the ratio of the transient to the diffusive terms. The parameters P and Pe take different values in each constituent.

l and L introduce two dimensionless (micro and macro) space variables $\mathbf{y} = \mathbf{X}/l$ and $\mathbf{x} = \mathbf{X}/L$, respectively. We use l to make the equations dimensionless. For the sake of simplicity, notations for dimensionless and dimensional quantities are left similar, except for space variables. The dimensionless equations can then be written in the form

– in Ω_k

$$\mathrm{P}\,\frac{\partial C_k T_k}{\partial t} = \frac{\partial}{\partial y_i}\left(\lambda_{k_{ij}}\frac{\partial T_k}{\partial y_j}\right) - \mathrm{Pe} V_{k_i}\,\frac{\partial C_k T_k}{\partial y_i}; \tag{4}$$

– on Γ_{kl}

$$\mathrm{N}_{\lambda_{kl}}\lambda_{k_{ij}}\left(\frac{\partial T_k}{\partial y_j}\right)\mathrm{N}_i = \lambda_{l_{ij}}\left(\frac{\partial T_1}{\partial y_j}\right)\mathrm{N}_i, \tag{5}$$

$$T_k = T_l \tag{6}$$

with

$$C_k = \rho_k C_{p_k} \quad \text{and} \quad \mathrm{N}_{\lambda_{kl}} = \frac{\lambda_k}{\lambda_l}.$$

3. Reference Values

Reference values have to be chosen. As an example, we consider the paper web characteristic reference values given in Table I.

Table I. Reference values for the physical properties of the porous medium components

s (J m^{-1} s^{-1} K^{-1})	**C$_{ps}$** (J kg^{-1} K^{-1})	s (kg m^{-3})	s (m^2 s^{-1})
0.33	1.33×10^3	1.5×10^3	1.6×10^{-7}
w	**C$_{pw}$**	**w**	**w**
0.602	4.18×10^3	10^3	1.4×10^{-7}
a	**C$_{pa}$**	**a**	**a**
0.026	10^3	1.23	2.1×10^{-5}

The scale ratio ε is defined as the ratio of l the characteristic microscopic size (pore) to the macroscopic one L (sample). For a paper web, the characteristic lengths can be estimated as $l = 10\ \mu\text{m}$ and $L = 1\ \text{mm}$. We thus obtain $\varepsilon \approx 10^{-2}$.

Some of the dimensionless numbers are immediately evaluated in term of ε:

$$\mathrm{N}_{\lambda_{\mathrm{aw}}} = 0.043 = \mathrm{O}(\varepsilon), \qquad \mathrm{N}_{\lambda_{\mathrm{sw}}} = 0.548 = \mathrm{O}(1),$$

$$P_{\mathrm{w}} = \frac{C_{\mathrm{w}} l^2}{\lambda_{\mathrm{w}} \tau} = \mathrm{O}(\varepsilon^{-1}\mathrm{P_a}), \qquad P_{\mathrm{s}} = \frac{C_{\mathrm{s}} l^2}{\lambda_{\mathrm{s}} \tau} = \mathrm{O}(\mathrm{P_w}).$$

For simplicity, we will denote now P the value common to P_{w} and P_{s}.

With the preceding data, we are left with two variable dimensionless numbers: the Péclet number Pe and the inverse P of the Fourier number.

Each physical quantity ϕ is then looked for in the form of asymptotic expansions with respect to the powers of ε:

$$\phi = \phi^{(0)} + \varepsilon \cdot \phi^{(2)} + \varepsilon^2 \cdot \phi^{(3)} + \cdots \tag{7}$$

4. The Different Macroscopic Descriptions

By applying the homogenisation process, we obtain macroscopic models that are approximations. The model accuracies are expressed in term of positive powers of ε. So, the more the scale separation is important, the better is the result.

We first consider in Section 4.1 small Péclet numbers, i.e., small convection. Then the Péclet number is increased (Section 4.2) and dispersion appears. We limit the presentation to the resulting macroscopic models. However, the details of the analysis in case of model B-1 (Section 4.2) are given in the Appendix. The macroscopic models are presented in dimensionless form. In the following, the first-order velocity term $\mathbf{V}^{(0)}$ is obtained separately from a classical Darcy problem, see Auriault (1991b) for details.

4.1. SMALL PÉCLET NUMBER: DIFFUSION-CONVECTION

A-1: $\mathrm{Pe} \leqslant \mathrm{O}(\varepsilon^2), \quad \mathrm{P} = \mathrm{O}(\varepsilon^2)$

The convection at the microscale is very weak, and does not appear at the macroscale. The time variation appears in the macroscale description. $|\Omega|$ stands for the period volume.

$$\langle C \rangle_{\mathrm{w,s}} \frac{\partial T}{\partial t} = \frac{\partial}{\partial x_i}\left(\lambda_{ij}^{\mathrm{eff}} \frac{\partial T}{\partial x_j}\right) + \mathrm{O}(\varepsilon), \tag{8}$$

with

$$\lambda_{ij}^{\mathrm{eff}} = \frac{1}{|\Omega|} \int_{\Omega_{\mathrm{w}}+\Omega_{\mathrm{s}}} \lambda_{ij} \left(I_{kj} + \frac{\partial \chi_{Ik}}{\partial y_j}\right) \mathrm{d}\Omega, \tag{9}$$

$$\langle C\rangle_{\mathrm{w,s}} = \frac{1}{|\Omega|}\left(\int_{\Omega_{\mathrm{w}}} C_{\mathrm{w}}\mathrm{d}\Omega + \int_{\Omega_{\mathrm{s}}} C_{\mathrm{s}}\,\mathrm{d}\Omega\right). \tag{10}$$

χ_{I} is given by a local boundary problem to be solved on the period Ω:

$$\frac{\partial}{\partial y_i}\left(\lambda_{\mathrm{a}_{ik}}\right) = -\frac{\partial}{\partial y_i}\left(\lambda_{\mathrm{a}_{ij}}\frac{\partial \chi_{\mathrm{Ia_{ik}}}}{\partial y_j}\right), \quad \text{in } \Omega_{\mathrm{a}}. \tag{11}$$

Similar equations are obtained in both Ω_{w} and Ω_{s}. On the boundaries, we have

$$0 = \lambda_{\mathrm{w}_{ij}}\left(\mathrm{I}_{jk} + \frac{\partial \chi_{\mathrm{Iw_k}}}{\partial y_j}\right)\mathrm{N}_i, \quad \text{on } \Gamma_{\mathrm{wa}}, \tag{12}$$

$$\lambda_{\mathrm{s}_{ij}}\left(\mathrm{I}_{jk} + \frac{\partial \chi_{\mathrm{Is_k}}}{\partial y_j}\right)\mathrm{N}_i = \lambda_{\mathrm{w}_{ij}}\left(\mathrm{I}_{jk} + \frac{\partial \chi_{\mathrm{Iw_k}}}{\partial y_j}\right)\mathrm{N}_i, \quad \text{on } \Gamma_{\mathrm{ws}}, \tag{13}$$

$$\chi_{\mathrm{Ia}} = \chi_{\mathrm{Iw}}, \quad \text{on } \Gamma_{\mathrm{wa}}, \tag{14}$$

$$\chi_{\mathrm{Is}} = \chi_{\mathrm{Iw}}, \quad \text{on } \Gamma_{\mathrm{ws}}, \tag{15}$$

with $\chi_{\mathrm{I}\alpha} = \chi_{\mathrm{I}\alpha}(y)$, $(\alpha = \mathrm{w, a, s})$.

$\chi_{\mathrm{I}\alpha}$ is determined to an added arbitrary constant, of no importance (see Equation (9)). Unicity is needed for numerical investigation. It can be obtained by prescribing an extraneous condition, $\langle\chi_{\mathrm{I}\alpha}\rangle_\alpha = 0$ or by fixing the value of $\chi_{\mathrm{I}\alpha}$ at any particular point in the pores. Moreover, χ_{I} may be calculated analytically for a simple geometry (cf. Auriault *et al.*, 1996). $^{\mathrm{eff}}$ is a classical effective thermal conductivity: $^{\mathrm{eff}}$ is the ratio between volume-averaged temperature gradients and heat flux (see annex).

A-2: $\mathrm{Pe} \leqslant \mathrm{O}(\varepsilon^2)$, $\quad \mathrm{P} \leqslant \mathrm{O}(\varepsilon^3)$

The transient term order is now weaker, and disappears at the macroscale level:

$$0 = \frac{\partial}{\partial x_i}\left(\lambda_{ij}^{\mathrm{eff}}\frac{\partial T}{\partial x_j}\right) + \mathrm{O}(\varepsilon). \tag{16}$$

The effective thermal conductivity is equivalent to the one presented in the case A-1.

A-3: $\mathrm{Pe} = \mathrm{O}(\varepsilon)$, $\quad \mathrm{P} = \mathrm{O}(\varepsilon^3)$

The convection, being increased at the microscale, appears now at the macroscale level:

$$\langle C\rangle_{\mathrm{w,s}}\frac{\partial T}{\partial t} = \frac{\partial}{\partial x_i}\left(\lambda_{ij}^{\mathrm{eff}}\frac{\partial T}{\partial x_j}\right) - \langle \mathrm{C}V_i\rangle_\Omega\frac{\partial T}{\partial x_i} + \mathrm{O}(\varepsilon). \tag{17}$$

We remark that due to the contrast of the physical properties, the transient term takes into account the solid and fluid phases only, contrary to the convective term.

A-4: $\mathrm{Pe} = \mathrm{O}(\varepsilon), \quad \mathrm{P} \leqslant \mathrm{O}(\varepsilon^3)$

The transient term disappears in the macroscopic description as in the case A-2, but the convective effect has to be introduced:

$$0 = \frac{\partial}{\partial x_i}\left(\lambda_{ij}^{\mathrm{eff}}\frac{\partial T}{\partial x_j}\right) - \langle C V_i\rangle_\Omega \frac{\partial T}{\partial x_i} + \mathrm{O}(\varepsilon). \tag{18}$$

A-5: $\mathrm{Pe}_{\mathrm{a,w}} = \mathrm{O}(\varepsilon),\ \mathrm{Pe}_{\mathrm{s}} = \mathrm{O}(\varepsilon^2), \quad \mathrm{P} = \mathrm{O}(\varepsilon^2)$

As an example, we may consider that the solid convective term is smaller than those for water and air, at the microscale. Consequently, it disappears from the macroscopic description:

$$\langle c\rangle_{\mathrm{w,s}}\frac{\partial T}{\partial t} = \frac{\partial}{\partial x_i}\left(\lambda_{ij}^{\mathrm{eff}}\frac{\partial T}{\partial x_j}\right) - \langle C V_i\rangle_{\Omega_{\mathrm{a}}+\Omega_{\mathrm{w}}}\frac{\partial T}{\partial x_i} + \mathrm{O}(\varepsilon). \tag{19}$$

Conclusion. Note that the richest model is model A-3. It is possible to show that there is a continuous passage from this model to the other ones. Notes also that the effective conductivity λ^{eff} is similar in all models considered in Section 4.1.

4.2. DISPERSION

In the following cases, the Péclet number is increased by an order of magnitude.

B-1: $\mathrm{Pe} = \mathrm{O}(1), \quad \mathrm{P} = \mathrm{O}(\varepsilon)$

Details of the homogenisation process are given in the Appendix.

Both the transient term and the convection exist at the microscopic level. A second transient term appears in the first term of the equation. The dispersion is also present, as can be seen from the effective thermal conductivity. The macroscopic description is given by (A.24).

$$\begin{aligned}(\langle C\rangle_{\Omega_{\mathrm{w}}+\Omega_{\mathrm{s}}} + \varepsilon\langle C\rangle_{\Omega_{\mathrm{a}}})\frac{\partial T}{\partial t} + \varepsilon\langle C\chi_{\mathrm{II}_i}\rangle_{\Omega_{\mathrm{w}}+\Omega_{\mathrm{s}}}\frac{\partial^2 T}{\partial t\,\partial x_i} \\ = \varepsilon\frac{\partial}{\partial x_i}\left(\lambda_{ij}^{**\mathrm{eff}}\frac{\partial T}{\partial x_j}\right) - \left(\langle C V_i\rangle_\Omega\frac{\partial T}{\partial x_i}\right) + \mathrm{O}(\varepsilon^2),\end{aligned} \tag{20}$$

with

$$\lambda_{ij}^{**\mathrm{eff}} = \frac{1}{|\Omega|}\int_{\Omega_{\mathrm{w,s}}}\left[\lambda_{ij}\left(I_{ij} + \frac{\partial\chi_{\mathrm{II}_j}}{\partial y_l}\right) - C^{(0)}V_i^{(0)}\chi_{\mathrm{II}_j}\right]\mathrm{d}\Omega \tag{21}$$

and where

$$\langle \mathbf{V}_i^{(0)}\rangle^* = \frac{\langle \mathbf{C}^{(0)}\mathbf{V}_i^{(0)}\rangle_\Omega}{\langle C\rangle_{\Omega_{\mathrm{w}}+\Omega_{\mathrm{s}}}} \quad (i = \mathrm{w, s})$$

χ_{II} (different from χ_{I}) is defined by the following local boundary problem to be solved on the period Ω:

$$C_{\mathrm{a}}^{(0)} V_{\mathrm{a}_k}^{(0)} = \frac{\partial}{\partial y_i} \left(\lambda_{\mathrm{a}_{ij}} \left(\mathrm{I}_{jk} + \frac{\partial \chi_{\mathrm{IIa}_k}}{\partial y_j} \right) \right) - C_{\mathrm{a}}^{(0)} V_{\mathrm{a}_i}^{(0)} \frac{\partial \chi_{\mathrm{IIa}_k}}{\partial y_i}, \quad \text{in } \Omega_{\mathrm{a}}, \tag{22}$$

$$C_{\mathrm{w}}^{(0)} (\langle V_{\mathrm{w}_k}^{(0)} \rangle^* - V_{\mathrm{w}_k}^{(0)})$$
$$= \frac{\partial}{\partial y_i} \left(\lambda_{\mathrm{w}_{ij}} \left(\mathrm{I}_{jk} + \frac{\partial \chi_{\mathrm{IIw}_k}}{\partial y_j} \right) \right) - C_{\mathrm{w}}^{(0)} V_{\mathrm{w}_i}^{(0)} \frac{\partial \chi_{\mathrm{IIw}_k}}{\partial y_i}, \quad \text{in } \Omega_{\mathrm{w}}, \tag{23}$$

$$C_{\mathrm{s}}^{(0)} (\langle V_{\mathrm{s}_k}^{(0)} \rangle^* - V_{\mathrm{s}_k}^{(0)})$$
$$= \frac{\partial}{\partial y_i} \left(\lambda_{\mathrm{s}_{ij}} \left(\mathrm{I}_{jk} + \frac{\partial \chi_{\mathrm{IIs}_k}}{\partial y_j} \right) \right) - C_{\mathrm{s}}^{(0)} V_{\mathrm{s}_i}^{(0)} \frac{\partial \chi_{\mathrm{IIs}_k}}{\partial y_i}, \quad \text{in } \Omega_{\mathrm{s}}, \tag{24}$$

$$0 = \lambda_{\mathrm{w}_{ij}} \left(\mathrm{I}_{jk} + \frac{\partial \chi_{\mathrm{IIw}_k}}{\partial y_j} \right) \mathrm{N}_i, \quad \text{on } \Gamma_{\mathrm{wa}}, \tag{25}$$

$$\lambda_{\mathrm{s}_{ij}} \left(\mathrm{I}_{jk} + \frac{\partial \chi_{\mathrm{IIs}_k}}{\partial y_j} \right) \mathrm{N}_i = \lambda_{\mathrm{w}_{ij}} \left(\mathrm{I}_{jk} + \frac{\partial \chi_{\mathrm{IIw}_k}}{\partial y_j} \right) \mathrm{N}_i, \quad \text{on } \Gamma_{\mathrm{ws}}, \tag{26}$$

$$\chi_{\mathrm{IIa}} = \chi_{\mathrm{IIw}}, \quad \text{on } \Gamma_{\mathrm{wa}}, \tag{27}$$

$$\chi_{\mathrm{IIs}} = \chi_{\mathrm{IIw}}, \quad \text{on } \Gamma_{\mathrm{ws}}, \tag{28}$$

with $\chi_{\mathrm{II}\alpha} = \chi_{\mathrm{II}\alpha}(y) \quad (\alpha = \mathrm{w, a, s})$.

The above boundary-value problem defines $\chi_{\mathrm{II}\alpha}$ to an arbitrary constant, that may be fixed using the condition $\langle \chi_{\mathrm{II}\alpha} \rangle_{\Omega_\alpha} = 0$.

$\lambda^{**\mathrm{eff}}$ can be proved to be a dispersive thermal conductivity. It depends on the Darcy velocity $\mathbf{V}^{(0)}$, where the exponent $^{(0)}$ represents the first-order approximation.

B-2: $\mathrm{Pe} = \mathrm{O}(1), \quad \mathrm{P} = \mathrm{O}(\varepsilon^2)$

Now, the transient term and the convection exist. Dispersion is also present:

$$\varepsilon \langle C \rangle_{\mathrm{w,s}} \frac{\partial T}{\partial t} = \varepsilon \frac{\partial}{\partial x_i} \left(\lambda_{ij}^{***\mathrm{eff}} \frac{\partial T}{\partial x_j} \right) - \langle C V \rangle_{\Omega} \frac{\partial T}{\partial x_i} + \mathrm{O}(\varepsilon^2), \tag{29}$$

with

$$\lambda_{ij}^{**\mathrm{eff}} = \frac{1}{|\Omega|} \int_{\Omega_{\mathrm{w,s}}} \left[\lambda_{ij} \left(I_{ij} + \frac{\partial \chi_{\mathrm{III}_j}}{\partial y_l} \right) - C^{(0)} V_i^{(0)} \chi_{\mathrm{III}_j} \right] \mathrm{d}\Omega. \tag{30}$$

$\lambda^{***\text{eff}}$ is also a dispersive thermal conductivity, different from $\lambda^{**\text{eff}}$. χ_{III} is determined by the following boundary problem to be solved:

$$C_{\text{a}}^{(0)} V_{\text{a}_k}^{(0)} = \frac{\partial}{\partial y_i} \left(\lambda_{\text{a}_{ij}} \left(\mathrm{I}_{jk} + \frac{\partial \chi_{\text{IIIa}_k}}{\partial y_j} \right) \right) - C_{\text{a}}^{(0)} V_{\text{a}_i}^{(0)} \frac{\partial \chi_{\text{IIIa}_k}}{\partial y_i}, \quad \text{in } \Omega_{\text{a}}, \tag{31}$$

$$C_{\text{w}}^{(0)} V_{\text{w}_k}^{(0)} = \frac{\partial}{\partial y_i} \left(\lambda_{\text{w}_{ij}} \left(\mathrm{I}_{jk} + \frac{\partial \chi_{\text{IIIw}_k}}{\partial y_j} \right) \right) - C_{\text{w}}^{(0)} V_{\text{w}_i}^{(0)} \frac{\partial \chi_{\text{IIIw}_k}}{\partial y_i}, \quad \text{in } \Omega_{\text{w}}, \tag{32}$$

$$C_{\text{s}}^{(0)} V_{\text{s}_k}^{(0)} = \frac{\partial}{\partial y_i} \left(\lambda_{\text{s}_{ij}} \left(\mathrm{I}_{jk} + \frac{\partial \chi_{\text{IIIs}_k}}{\partial y_j} \right) \right) - C_{\text{s}}^{(0)} V_{\text{s}_i}^{(0)} \frac{\partial \chi_{\text{IIIs}_k}}{\partial y_i}, \quad \text{in } \Omega_{\text{s}}, \tag{33}$$

$$0 = \lambda_{\text{w}_{ij}} \left(\mathrm{I}_{jk} + \frac{\partial \chi_{\text{IIIw}_k}}{\partial y_j} \right) \mathrm{N}_i, \quad \text{on } \Gamma_{\text{wa}}, \tag{34}$$

$$\lambda_{\text{s}_{ij}} \left(\mathrm{I}_{jk} + \frac{\partial \chi_{\text{IIIs}_k}}{\partial y_j} \right) \mathrm{N}_i = \lambda_{\text{w}_{ij}} \left(\mathrm{I}_{jk} + \frac{\partial \chi_{\text{IIIw}_k}}{\partial y_j} \right) \mathrm{N}_i, \quad \text{on } \Gamma_{\text{ws}}, \tag{35}$$

$$\chi_{\text{IIIa}} = \chi_{\text{IIIw}}, \quad \text{on } \Gamma_{\text{wa}}, \tag{36}$$

$$\chi_{\text{IIIs}} = \chi_{\text{IIIw}}, \quad \text{on } \Gamma_{\text{ws}}, \tag{37}$$

with $\chi_{\text{II}\alpha} = \chi_{\text{II}\alpha}(y) \quad (\alpha = \text{w, a, s})$.

χ_{III} is determined to an arbitrary constant.

B-3: $\text{Pe} = \mathrm{O}(1), \quad \mathrm{P} \leqslant \mathrm{O}(\varepsilon^3)$

Now, the transient term disappears in the macroscopic description

$$0 = \varepsilon \frac{\partial}{\partial x_i} \left(\lambda_{ij}^{***\text{eff}} \frac{\partial T}{\partial x_j} \right) - \langle CV \rangle_\Omega \frac{\partial T}{\partial x_i} + \mathrm{O}(\varepsilon^2). \tag{38}$$

Conclusion. Considering all the models B, model B-1 appears as the richest one. It is analysed in details in the Appendix.

The cases with $\text{Pe} = \mathrm{O}(\varepsilon^{-1})$ are not homogenisable. It means that no equivalent macroscopic description exists. A macroscopic description can be obtained by a mathematical technique different from the multiple scale separation method, e.g., the volume averaging method. However, in such a case, the obtained macroscopic model is dependent on the macroscopic sample size and on the applied macroscopic boundary conditions. That is to say, such a macroscopic model is not intrinsic to the studied materials.

5. Example of the Model Catalogue Use: Hot Pressing of a Paper Web

As an example, we consider the hot pressing of a paper web, see Figure 2.

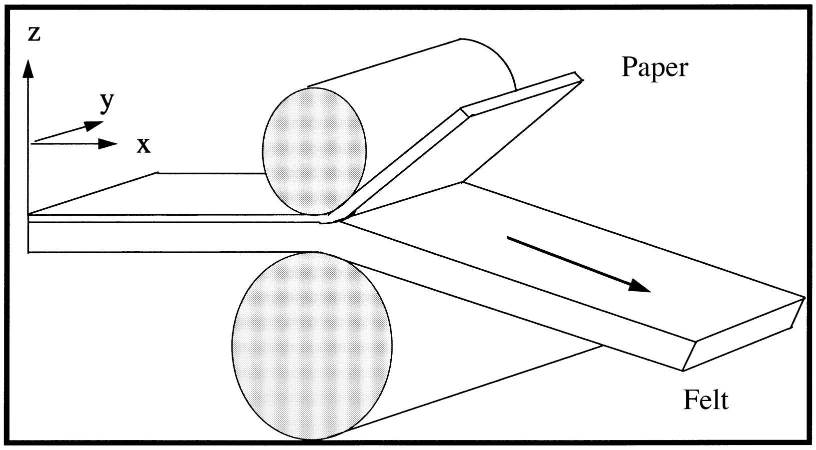

Figure 2. Schematic view of a press section.

The parameters P and Pe can be evaluated as follows. We consider a constant web velocity during the process. Therefore, using Euler variables, the time-dependence disappears and $\mathrm{P} = 0$. Furthermore, from the characteristic values in Table I and a Darcy velocity equal to $10^{-2}\ \mathrm{m\,s^{-1}}$, we obtain the Péclet number as

$$\mathrm{Pe} = \frac{\rho C_{\mathrm{p}} V l}{\lambda} \# \frac{10^3 \times 10^3 \times 10^{-2} \times 10^{-6}}{0.5} = \mathrm{O}(10^{-2}) = \mathrm{O}(\varepsilon).$$

Consequently, the adopted model for the hot pressing numerical modelling should be model A-4:

$$0 = \frac{\partial}{\partial x_i}\left(\lambda_{ij}^{\mathrm{eff}} \frac{\partial T}{\partial x_j}\right) - \langle C V_i \rangle_\Omega \frac{\partial T}{\partial x_i}. \tag{39}$$

Due to the complexity of the pore geometry, the effective conductivity λ^{eff} is generally difficult to determine. Therefore, three complementary hypotheses may be adopted to simplify the model:

– The first one concerns the thermal effective conductivities of the fluid and the solid. They are of the same order of magnitude, $\lambda_{\mathrm{w}} = \mathrm{O}(\lambda_{\mathrm{s}})$. For simplicity, they can be assumed as isotropic and constant. Therefore, Equation (39) becomes

$$0 = \frac{\partial}{\partial x_i}\left(\lambda_{\mathrm{w}}\,(n_{\mathrm{w}} + n_{\mathrm{s}})\,\mathrm{I}_{ij} \frac{\partial T}{\partial x_i}\right) - \langle C V_i \rangle_\Omega \frac{\partial T}{\partial x_i}, \tag{40}$$

where n_k ($k = \mathrm{w, s}$) is the volume ratio and $\mathbf{I}$ is the identity matrix.

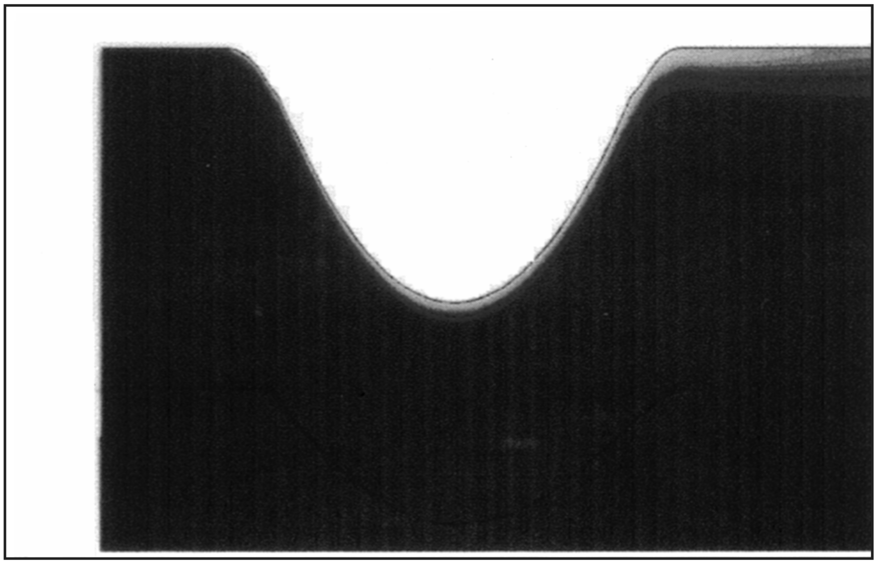

Figure 3. Temperature field in the paper during hot pressing: an example.

– In a paper web pressing process, the gas volume is generally very small in comparison to the solid and liquid ones: $n_s + n_w \# 1$. Equation (40) may therefore be written in the form

$$0 = \frac{\partial}{\partial x_i}\left(\lambda_w \mathrm{I}_{ij}\,\frac{\partial T}{\partial x_i}\right) - \langle CV_i\rangle_\Omega\,\frac{\partial T}{\partial x_i}. \tag{41}$$

– Finally, the contribution of the air in the convective term can be assumed to be negligible: $\rho_a C_{pa} V_a \ll \rho_w C_{pw} V_w$, then Equation (41) simplifies to

$$0 = \frac{\partial}{\partial x_i}\left(\lambda_w\ \mathrm{I}_{ij}\,\frac{\partial T}{\partial x_i}\right) - \langle CV_i\rangle_{\Omega_s+\Omega_w}\,\frac{\partial T}{\partial x_i}. \tag{42}$$

Equation (42) was used for the numerical modelling of the paper web hot pressing (Bloch, 1995). An example of a numerical result is shown in Figure 3.

Note that Equation (42) has been obtained, *after* determining the correct structure of the energy equation and then introducing simplifying assumptions.

6. Conclusion

Despite the physics at the pore scale being similar, we obtain different law structures with different effective coefficients for the equivalent medium. In practical cases, the right macroscopic model is determined by evaluating the different dimensionless numbers, in function of the scale factor ε (defined as the ratio of the microscopic characteristic dimension to the macroscopic one). Then a convenient model is found in the presented catalogue. There is a continuous passage from the most appropriate model to the others. It is necessary to underline the importance of the ε factor, because the evaluation of the dimensionless parameters will depend on its value.

Note that, in the case of very large Péclet number, the medium cannot be homogenised. That means that the experimentally determined macroscopic properties are not intrinsic to the media but are specific to the boundary problem considered in the experiment. One should then be very careful when using macroscopic models, which are shown in the paper to be valid, in restricted ranges. Finally, note that the presented results are applicable to any porous medium that satisfies the microscopic hypotheses in use here.

Appendix: Model B-1: $\mathbf{Pe} = \mathrm{O}(1)$ $\mathbf{P} = \mathrm{O}(\varepsilon)$

By using e to make dimensionless the local equations, we obtain

$$\varepsilon^2 C_\mathrm{a} \frac{\partial T_\mathrm{a}}{\partial t} = \frac{\partial}{\partial y_i}\left(\lambda_{\mathrm{a}_{ij}} \frac{\partial T_\mathrm{a}}{\partial_{y_j}}\right) - C_\mathrm{a} V_{\mathrm{a}_i} \frac{\partial T_\mathrm{a}}{\partial y_i}, \quad \text{in } \Omega_\mathrm{a}, \tag{A.1}$$

$$\varepsilon C_\mathrm{w} \frac{\partial T_\mathrm{w}}{\partial t} = \frac{\partial}{\partial y_i}\left(\lambda_{\mathrm{w}_{ij}} \frac{\partial T_\mathrm{w}}{\partial y_j}\right) - C_\mathrm{w} V_{\mathrm{w}_i} \frac{\partial T_\mathrm{w}}{\partial y_i}, \quad \text{in } \Omega_\mathrm{w}, \tag{A.2}$$

$$\varepsilon C_\mathrm{s} \frac{\partial T_\mathrm{s}}{\partial t} = \frac{\partial}{\partial y_i}\left(\lambda_{\mathrm{s}_{ij}} \frac{\partial T_\mathrm{s}}{\partial y_j}\right) - C_\mathrm{s} V_{\mathrm{s}_i} \frac{\partial T_\mathrm{s}}{\partial y_i}, \quad \text{in } \Omega_\mathrm{s}, \tag{A.3}$$

$$\varepsilon \lambda_{\mathrm{a}_{ij}} \left(\frac{\partial T_\mathrm{a}}{\partial y_j}\right) N_i = \lambda_{\mathrm{w}_{ij}} \left(\frac{\partial T_\mathrm{w}}{\partial y_j}\right) N_i, \quad \text{on } \Gamma_\mathrm{wa}, \tag{A.4}$$

$$\lambda_{\mathrm{s}_{ij}} \left(\frac{\partial T_\mathrm{s}}{\partial y_j}\right) N_i = \lambda_{\mathrm{w}_{ij}} \left(\frac{\partial T_\mathrm{w}}{\partial y_j}\right) N_i \quad \text{on } \Gamma_\mathrm{ws}, \tag{A.5}$$

$$T_\mathrm{a} = T_\mathrm{w} \quad \text{on } \Gamma_\mathrm{wa}, \tag{A.6}$$

$$T_\mathrm{s} = T_\mathrm{w} \quad \text{on } \Gamma_\mathrm{ws}. \tag{A.7}$$

Introducing expansions like (7) successively yields

in Ω_a

$$\frac{\partial}{\partial y_i}\left(\lambda_{\mathrm{a}_{ij}} \frac{\partial T_\mathrm{a}^{(0)}}{\partial y_j}\right) - C_\mathrm{a}^{(0)} V_{\mathrm{a}_i}^{(0)} \frac{\partial T_\mathrm{a}^{(0)}}{\partial y_i} = 0 \tag{A.1.0}$$

$$0 = \frac{\partial}{\partial y_i}\left(\lambda_{\mathrm{a}_{ij}} \left(\frac{\partial T_\mathrm{a}^{(1)}}{\partial y_j} + \frac{\partial T_\mathrm{a}^{(0)}}{\partial x_j}\right)\right) + \frac{\partial}{\partial x_i}\left(\lambda_{\mathrm{a}_{ij}} \frac{\partial T_\mathrm{a}^{(0)}}{\partial y_j}\right) -$$

$$- C_{\mathrm{a}}^{(0)} V_{\mathrm{a}_i}^{(0)} \frac{\partial T_{\mathrm{a}}^{(1)}}{\partial y_i} - C_{\mathrm{a}}^{(0)} V_{\mathrm{a}_i}^{(1)} \frac{\partial T_{\mathrm{a}}^{(0)}}{\partial y_i} - C_{\mathrm{a}}^{(0)} V_{\mathrm{a_i}}^{(0)} \frac{\partial T_{\mathrm{a}}^{(0)}}{\partial x_i} - $$

$$- C_{\mathrm{a}}^{(1)} V_{\mathrm{a}_i}^{(0)} \frac{\partial T_{\mathrm{a}}^{(0)}}{\partial y_i} \tag{A.1.1}$$

$$\begin{aligned}
C_{\mathrm{a}}^{(0)} \frac{\partial T_{\mathrm{a}}^{(0)}}{\partial t} = {} & \frac{\partial}{\partial y_i}\left(\lambda_{\mathrm{a}_{ij}}\left(\frac{\partial T_{\mathrm{a}}^{(2)}}{\partial y_j} + \frac{\partial T_{\mathrm{a}}^{(1)}}{\partial x_j}\right)\right) + \\
& + \frac{\partial}{\partial x_i}\left(\lambda_{\mathrm{a}_{ij}}\left(\frac{\partial T_{\mathrm{a}}^{(1)}}{\partial y_j} + \frac{\partial T_{\mathrm{a}}^{(0)}}{\partial x_j}\right)\right) - \\
& - C_{\mathrm{a}}^{(0)}\left(V_{\mathrm{a}_i}^{(0)} \frac{\partial T_{\mathrm{a}}^{(2)}}{\partial y_i} + V_{\mathrm{a}_i}^{(1)} \frac{\partial T_{\mathrm{a}}^{(1)}}{\partial y_i} + V_{\mathrm{a}_i}^{(2)} \frac{\partial T_{\mathrm{a}}^{(0)}}{\partial y_i}\right) - \\
& - C_{\mathrm{a}}^{(0)}\left(V_{\mathrm{a}_i}^{(0)} \frac{\partial T_{\mathrm{a}}^{(1)}}{\partial x_i} + V_{\mathrm{a}_i}^{(1)} \frac{\partial T_{\mathrm{a}}^{(0)}}{\partial x_i}\right) - C_{\mathrm{a}}^{(1)} V_{\mathrm{a}_i}^{(0)} \frac{\partial T_{\mathrm{a}}^{(0)}}{\partial x_i} - \\
& - C_{\mathrm{a}}^{(1)}\left(V_{\mathrm{a}_i}^{(0)} \frac{\partial T_{\mathrm{a}}^{(1)}}{\partial y_i} + V_{\mathrm{a}_i}^{(1)} \frac{\partial T_{\mathrm{a}}^{(0)}}{\partial y_i}\right) - C_{\mathrm{a}}^{(2)} V_{\mathrm{a}_i}^{(0)} \frac{\partial T_{\mathrm{a}}^{(0)}}{\partial y_i}.
\end{aligned} \tag{A.1.2}$$

in Ω_{w}

$$\frac{\partial}{\partial y_i}\left(\lambda_{\mathrm{w}_{ij}} \frac{\partial T_{\mathrm{w}}^{(0)}}{\partial y_j}\right) - C_{\mathrm{w}}^{(0)} V_{\mathrm{w}_i}^{(0)} \frac{\partial T_{\mathrm{w}}^{(0)}}{\partial y_i} = 0, \tag{A.2.0}$$

$$\begin{aligned}
C_{\mathrm{w}}^{(0)} \frac{\partial T_{\mathrm{w}}^{(0)}}{\partial t} = {} & \frac{\partial}{\partial y_i}\left(\lambda_{\mathrm{w}_{ij}}\left(\frac{\partial T_{\mathrm{w}}^{(1)}}{\partial y_j} + \frac{\partial T_{\mathrm{w}}^{(0)}}{\partial x_j}\right)\right) + \frac{\partial}{\partial x_i}\left(\lambda_{\mathrm{w}_{ij}} \frac{\partial T_{\mathrm{w}}^{(0)}}{\partial y_j}\right) - \\
& - C_{\mathrm{w}}^{(0)}\left(V_{\mathrm{w}_i}^{(0)} \frac{\partial T_{\mathrm{w}}^{(1)}}{\partial y_i} + V_{\mathrm{w}_i}^{(1)} \frac{\partial T_{\mathrm{w}}^{(0)}}{\partial y_i}\right) - C_{\mathrm{w}}^{(0)} V_{\mathrm{w}_i}^{(0)} \frac{\partial T_{\mathrm{w}}^{(0)}}{\partial x_i} - \\
& - C_{\mathrm{w}}^{(1)} V_{\mathrm{w}_i}^{(0)} \frac{\partial T_{\mathrm{w}}^{(0)}}{\partial y_i},
\end{aligned} \tag{A.2.1}$$

$$
\begin{aligned}
& C_{\mathrm{w}}^{(0)} \frac{\partial T_{\mathrm{w}}^{(1)}}{\partial t} + C_{\mathrm{w}}^{(1)} \frac{\partial T_{\mathrm{w}}^{(0)}}{\partial t} \\
&\quad = \frac{\partial}{\partial y_i}\left(\lambda_{\mathrm{w}_{ij}}\left(\frac{\partial T_{\mathrm{w}}^{(2)}}{\partial y_j} + \frac{\partial T_{\mathrm{w}}^{(1)}}{\partial x_j}\right)\right) + \frac{\partial}{\partial x_i}\left(\lambda_{\mathrm{w}_{ij}}\left(\frac{\partial T_{\mathrm{w}}^{(1)}}{\partial y_j} + \frac{\partial T_{\mathrm{w}}^{(0)}}{\partial x_j}\right)\right) - \\
&\qquad - C_{\mathrm{w}}^{(0)}\left(V_{\mathrm{w}_i}^{(0)} \frac{\partial T_{\mathrm{w}}^{(2)}}{\partial y_i} + V_{\mathrm{w}_i}^{(2)} \frac{\partial T_{\mathrm{w}}^{(0)}}{\partial y_i} + V_{\mathrm{w}_i}^{(1)} \frac{\partial T_{\mathrm{w}}^{(1)}}{\partial y_i}\right) - \\
&\qquad - C_{\mathrm{w}}^{(0)}\left(V_{\mathrm{w}_i}^{(0)} \frac{\partial T_{\mathrm{w}}^{(1)}}{\partial x_i} + V_{\mathrm{w}_i}^{(1)} \frac{\partial T_{\mathrm{w}}^{(0)}}{\partial x_i}\right) - \left[C_{\mathrm{w}}^{(1)}\left(V_{\mathrm{w}_i}^{(0)} \frac{\partial T_{\mathrm{w}}^{(1)}}{\partial y_i} + V_{\mathrm{w}_i}^{(1)} \frac{\partial T_{\mathrm{w}}^{(0)}}{\partial y_i}\right)\right] - \\
&\qquad - C_{\mathrm{w}}^{(1)} V_{\mathrm{w}_i}^{(0)} \frac{\partial T_{\mathrm{w}}^{(0)}}{\partial x_i} - C_{\mathrm{w}}^{(2)} V_{\mathrm{w}_i}^{(0)} \frac{\partial T_{\mathrm{w}}^{(0)}}{\partial y_i},
\end{aligned}
\tag{A.2.2}
$$

in Ω_{s}

$$
\frac{\partial}{\partial y_i}\left(\lambda_{\mathrm{s}_{ij}} \frac{\partial T_{\mathrm{s}}^{(0)}}{\partial y_j}\right) - C_{\mathrm{s}}^{(0)} V_{\mathrm{s}_i}^{(0)} \frac{\partial T_{\mathrm{s}}^{(0)}}{\partial y_i} = 0,
\tag{A.3.0}
$$

$$
\begin{aligned}
C_{\mathrm{s}}^{(0)} \frac{\partial T_{\mathrm{s}}^{(0)}}{\partial t} &= \frac{\partial}{\partial y_i}\left(\lambda_{\mathrm{s}_{ij}}\left(\frac{\partial T_{\mathrm{s}}^{(1)}}{\partial y_j} + \frac{\partial T_{\mathrm{s}}^{(0)}}{\partial x_j}\right)\right) + \frac{\partial}{\partial x_i}\left(\lambda_{\mathrm{s}_{ij}} \frac{\partial T_{\mathrm{s}}^{(0)}}{\partial y_j}\right) - \\
&\quad - C_{\mathrm{s}}^{(0)}\left(V_{\mathrm{s}_i}^{(0)} \frac{\partial T_{\mathrm{s}}^{(1)}}{\partial y_i} + V_{\mathrm{s}_i}^{(1)} \frac{\partial T_{\mathrm{s}}^{(0)}}{\partial y_i}\right) - \\
&\quad - C_{\mathrm{s}}^{(0)} V_{\mathrm{s}_i}^{(0)} \frac{\partial T_{\mathrm{s}}^{(0)}}{\partial x_i} - C_{\mathrm{s}}^{(1)} V_{\mathrm{s}_i}^{(0)} \frac{\partial T_{\mathrm{s}}^{(0)}}{\partial y_i},
\end{aligned}
\tag{A.3.1}
$$

$$
\begin{aligned}
& C_{\mathrm{s}}^{(0)} \frac{\partial T_{\mathrm{s}}^{(1)}}{\partial t} + C_{\mathrm{s}}^{(1)} \frac{\partial T_{\mathrm{s}}^{(0)}}{\partial t} \\
&\quad = \frac{\partial}{\partial y_i}\left(\lambda_{\mathrm{s}_{ij}}\left(\frac{\partial T_{\mathrm{s}}^{(2)}}{\partial y_j} + \frac{\partial T_{\mathrm{s}}^{(1)}}{\partial x_j}\right)\right) + \frac{\partial}{\partial x_i}\left(\lambda_{\mathrm{s}_{ij}}\left(\frac{\partial T_{\mathrm{s}}^{(1)}}{\partial y_j} + \frac{\partial T_{\mathrm{s}}^{(0)}}{\partial x_j}\right)\right) - \\
&\qquad - C_{\mathrm{s}}^{(0)}\left(V_{\mathrm{s}_i}^{(0)} \frac{\partial T_{\mathrm{s}}^{(2)}}{\partial y_i} + V_{\mathrm{s}_i}^{(2)} \frac{\partial T_{\mathrm{s}}^{(0)}}{\partial y_i} + V_{\mathrm{s}_i}^{(1)} \frac{\partial T_{\mathrm{s}}^{(1)}}{\partial y_i}\right) - \\
&\qquad - C_{\mathrm{s}}^{(0)}\left(V_{\mathrm{s}_i}^{(0)} \frac{\partial T_{\mathrm{s}}^{(1)}}{\partial x_i} + V_{\mathrm{s}_i}^{(1)} \frac{\partial T_{\mathrm{s}}^{(0)}}{\partial x_i}\right) - C_{\mathrm{s}}^{(1)}\left(V_{\mathrm{s}_i}^{(0)} \frac{\partial T_{\mathrm{s}}^{(1)}}{\partial y_i} + V_{\mathrm{s}_i}^{(1)} \frac{\partial T_{\mathrm{s}}^{(0)}}{\partial y_i}\right) \\
&\qquad - C_{\mathrm{s}}^{(1)} V_{\mathrm{s}_i}^{(0)} \frac{\partial T_{\mathrm{s}}^{(0)}}{\partial x_i} - C_{\mathrm{s}}^{(2)} V_{\mathrm{s}_i}^{(0)} \frac{\partial T_{\mathrm{s}}^{(0)}}{\partial y_i},
\end{aligned}
\tag{A.3.2}
$$

on Γ_{wa}:

$$0 = \lambda_{\text{w}_{ij}} \left(\frac{\partial T_{\text{w}}^{(0)}}{\partial y_j} \right) N_i, \tag{A.4.0}$$

$$\lambda_{\text{a}_{ij}} \frac{\partial T_{\text{a}}^{(0)}}{\partial y_j} N_i = \lambda_{\text{w}_{ij}} \left(\frac{\partial T_{\text{w}}^{(1)}}{\partial y_{ij}} + \frac{\partial T_{\text{w}}^{(0)}}{\partial x_j} \right) N_i, \tag{A.4.1}$$

$$\lambda_{\text{a}_{ij}} \left(\frac{\partial T_{\text{a}}^{(1)}}{\partial y_j} + \frac{\partial T_{\text{a}}^{(0)}}{\partial x_j} \right) N_i = \lambda_{\text{w}_{ij}} \left(\frac{\partial T_{\text{w}}^{(2)}}{\partial y_j} + \frac{\partial T_{\text{w}}^{(1)}}{\partial x_j} \right) N_i, \tag{A.4.2}$$

on Γ_{ws}:

$$\lambda_{\text{s}_{ij}} \left(\frac{\partial T_{\text{s}}^{(0)}}{\partial y_j} \right) N_i = \lambda_{\text{w}_{ij}} \left(\frac{\partial T_{\text{w}}^{(0)}}{\partial y_j} \right) N_i, \tag{A.5.0}$$

$$\lambda_{\text{s}_{ij}} \left(\frac{\partial T_{\text{s}}^{(1)}}{\partial y_j} + \frac{\partial T_{\text{s}}^{(0)}}{\partial x_j} \right) N_i = \lambda_{\text{w}_{ij}} \left(\frac{\partial T_{\text{w}}^{(1)}}{\partial y_j} + \frac{\partial T_{\text{w}}^{(0)}}{\partial x_j} \right) N_i, \tag{A.5.1}$$

$$\lambda_{\text{s}_{ij}} \left(\frac{\partial T_{\text{s}}^{(2)}}{\partial y_j} + \frac{\partial T_{\text{s}}^{(1)}}{\partial x_j} \right) N_i = \lambda_{\text{w}_{ij}} \left(\frac{\partial T_{\text{w}}^{(2)}}{\partial y_{ij}} + \frac{\partial T_{\text{w}}^{(1)}}{\partial x_j} \right) N_i, \tag{A.5.2}$$

on Γ_{wa}:

$$T_{\text{a}}^{(0)} = T_{\text{w}}^{(0)}, \tag{A.6.0}$$

$$T_{\text{a}}^{(1)} = T_{\text{w}}^{(1)}, \tag{A.6.1}$$

$$T_{\text{a}}^{(2)} = T_{\text{w}}^{(2)}, \tag{A.6.2}$$

on Γ_{ws}:

$$T_{\text{s}}^{(0)} = T_{\text{w}}^{(0)}, \tag{A.7.0}$$

$$T_{\text{s}}^{(1)} = T_{\text{w}}^{(1)}, \tag{A.7.1}$$

$$T_{\text{s}}^{(2)} = T_{\text{w}}^{(2)}. \tag{A.7.2}$$

First boundary-value problem

As a first boundary-value problem of unknown $T^{(0)}$, we consider Equations (A.2.0), (A.3.0), the fluid incompressibility at the first order ($\mathrm{div}_y(\rho^{(0)}\mathbf{v}^{(0)}) = 0$), Equations (A.4.0) (A.5.0) and (A.7.0), and finally Equation (A.1.0) on Ω_a together with Equation (A.6.0). It is possible to show that

$$T_{\mathrm{w}}^{(0)} = T_{\mathrm{s}}^{(0)} = T_{\mathrm{a}}^{(0)} = T^{(0)}(x) \tag{A.8}$$

Second boundary-value problem

The second boundary-value problem is for the unknown $T^{(1)}$. Integrating Equations (A.2.1) and (A.3.1) on domains $\Omega_\alpha(\alpha = \mathrm{w, s})$, respectively, adding, member to member, the resulting equations and using the divergence theorem with the conditions (A.5.1) and (A.4.1) yields

$$\left\langle C^{(0)}\right\rangle_{\Omega_{\mathrm{w}}+\Omega_{\mathrm{s}}} \frac{\partial T^{(0)}}{\partial t} = -\left\langle C^{(0)} V_i^{(0)}\right\rangle_{\Omega} \frac{\partial T^{(0)}}{\partial x_i} \tag{A.9}$$

with

$$\langle C\rangle_{\Omega_{\mathrm{w}}+\Omega_{\mathrm{s}}} = \frac{1}{|\Omega|}\left(\int_{\Omega\mathrm{w}} C_{\mathrm{w}}\mathrm{d}\Omega + \int_{\Omega\mathrm{s}} C_{\mathrm{s}}\mathrm{d}\Omega\right).$$

We now investigate $T^{(1)}$ successively in domains Ω_{w} and Ω_{s} and next in domain Ω_a. For water and solid, Equation (A.2.1) or Equation (A.3.1) may be rewritten in the form

$$\begin{aligned} C_\alpha^{(0)} \frac{\partial T^{(0)}}{\partial t} \;=\; & \frac{\partial}{\partial y_i}\left(\lambda_{\alpha_{ij}}\left(\frac{\partial T_\alpha^{(1)}}{\partial y_j} + \frac{\partial T^{(0)}}{\partial x_j}\right)\right) + \frac{\partial}{\partial x_i}\left(\lambda_{ij}\frac{\partial T^{(0)}}{\partial y_j}\right) - \\ & - C_\alpha^{(0)}\left(V_{\alpha_i}^{(0)}\frac{\partial T_\alpha^{(1)}}{\partial y_i} + V_{\alpha_i}^{(1)}\frac{\partial T^{(0)}}{\partial y_i}\right) - \\ & - C_\alpha^{(0)} V_{\alpha_i}^{(0)}\frac{\partial T^{(0)}}{\partial x_i} - C_\alpha^{(1)} V_{\alpha_i}^{(0)}\frac{\partial T^{(0)}}{\partial y_i} \qquad (\alpha = \mathrm{w, s}). \end{aligned} \tag{A.10}$$

Then, using Equation (A.8) yields

$$\begin{aligned} C_\alpha^{(0)} \frac{\partial T_\alpha^{(1)}}{\partial t} \;=\; & \frac{\partial}{\partial y_i}\left(\lambda_{\alpha_{ij}}\left(\frac{\partial T_\alpha^{(1)}}{\partial y_j} + \frac{\partial T^{(0)}}{\partial x_j}\right)\right) - \\ & - C_\alpha^{(0)}\left(V_{\alpha_i}^{(0)}\frac{\partial T_\alpha^{(1)}}{\partial y_i} - V_{\alpha_i}^{(0)}\frac{\partial T^{(0)}}{\partial x_i}\right). \end{aligned} \tag{A.11}$$

Moreover, Equation (A.9) may be rewritten in the form

$$\frac{\partial T^{(0)}}{\partial t} = -\left\langle V_i^{(0)}\right\rangle^* \frac{\partial T^{(0)}}{\partial_{x_i}} \tag{A.12}$$

with

$$\langle V_i^{(0)}\rangle^* = \frac{\langle C^{(0)} V_i^{(0)}\rangle_\Omega}{\langle C\rangle_{\Omega_\mathrm{w}+\Omega_\mathrm{s}}}.$$

Hence we get

$$\begin{aligned}&\frac{\partial}{\partial y_i}\left(\lambda_{\alpha_{ij}}\left(\frac{\partial T_\alpha^{(1)}}{\partial y_j}+\frac{\partial T^{(0)}}{\partial x_j}\right)\right) - C_\alpha^{(0)} V_{\alpha_i}^{(0)} \frac{\partial T^{(1)}}{\partial y_i}\\ &\quad = C_\alpha^{(0)}\left(V_{\alpha_i}^{(0)} - \langle V_{\alpha_i}^{(0)}\rangle^*\right)\frac{\partial T^{(0)}}{\partial x_i}.\end{aligned} \tag{A.13}$$

It is possible to show that the boundary-value problem for $T_\alpha^{(1)}, \alpha = \mathrm{w, a, s}$, has a unique solution. Because of the linearity, it can be put in the form

$$T_\alpha^{(1)} = \chi_{\alpha_{\mathrm{II}}\ i}\frac{\partial T_\alpha^{(0)}}{\partial x_i} + \overline{T}_\alpha(x,t) \quad \alpha = \mathrm{w, s}. \tag{A.14}$$

The boundary-value problem for χ_{II} is given above by Equations (22)–(28).

Considering now the gas (air), Equation (A.1.1) can be rewritten in the form

$$\begin{aligned}0 = {}&\frac{\partial}{\partial y_i}\left(\lambda_{\mathrm{a}_{ij}}\left(\frac{\partial T_\mathrm{a}^{(1)}}{\partial y_j}+\frac{\partial T_\mathrm{a}^{(0)}}{\partial x_j}\right)\right) + \frac{\partial}{\partial x_i}\left(\lambda_{\mathrm{a}_{ij}}\frac{\partial T_\mathrm{a}^{(0)}}{\partial y_j}\right) - \\ &- C_\mathrm{a}^{(0)}\left(V_{\mathrm{a}_i}^{(0)}\frac{\partial T_\mathrm{a}^{(1)}}{\partial y_i} + V_{\mathrm{a}_i}^{(1)}\frac{\partial T_\mathrm{a}^{(0)}}{\partial y_i}\right) - C_\mathrm{a}^{(0)} V_{\mathrm{a}_i}^{(0)}\frac{\partial T_\mathrm{a}^{(0)}}{\partial x_i} - \\ &- C_\mathrm{a}^{(1)} V_{\mathrm{a}_i}^{(0)}\frac{\partial T_\mathrm{a}^{(0)}}{\partial y_i}.\end{aligned} \tag{A.15}$$

Using Equation (B.8) gives

$$0 = \frac{\partial}{\partial y_i}\left(\lambda_{\mathrm{a}_{ij}}\left(\frac{\partial T_\mathrm{a}^{(1)}}{\partial y_j}+\frac{\partial T_\mathrm{a}^{(0)}}{\partial x_j}\right)\right) - C_\mathrm{a}^{(0)}\left(V_{\mathrm{a}_i}^{(0)}\frac{\partial T_\mathrm{a}^{(1)}}{\partial y_i} - V_{\mathrm{a}_i}^{(0)}\frac{\partial T_\mathrm{a}^{(0)}}{\partial x_i}\right). \tag{A.16}$$

The boundary-value problem for $T_\mathrm{a}^{(1)}$ has a unique solution. Its linearity yields $T_\mathrm{a}^{(1)}$ in the form

$$T_\mathrm{a}^{(1)} = \chi_{\mathrm{a}_{\mathrm{II}}\ i}\frac{\partial T_\mathrm{a}^{(0)}}{\partial x_i} + \overline{T}_\mathrm{a}(x,t). \tag{A.17}$$

Consequently, we have for each component of the porous material

$$T_\alpha^{(1)} = \chi_{\alpha_{\mathrm{II}}\ i}\frac{\partial T_\alpha^{(0)}}{\partial x_i} + \overline{T}_\alpha(x,t) \quad \alpha = (\mathrm{w, s, a}). \tag{A.18}$$

Third boundary-value problem

We achieve the process by considering the third boundary-value problem of unknown $T^{(2)}$. Integrating Equations (A.2.2) and (A.3.2) on their respective domains of definition and using successively the divergence theorem applied to (A.7.2), (A.6.2), (A.5.2), (A.4.2) and (A.1.1), yield

$$\frac{\partial \left\langle C^{(0)} T^{(1)} + C^{(1)} T^{(0)} \right\rangle_{\Omega_\mathrm{w}+\Omega_\mathrm{s}}}{\partial t} + \left\langle C^{(0)} \right\rangle_{\Omega_\mathrm{a}} \frac{\partial T^{(0)}}{\partial t} = \frac{\partial}{\partial x_i} \left(\lambda^*_{ij} \frac{\partial T^{(0)}}{\partial x_j} \right) - \\ - \left(\left\langle C^{(0)} V_i^{(0)} \frac{\partial T^{(1)}}{\partial x_i} \right\rangle + \left\langle C^{(0)} V_i^{(1)} \right\rangle \frac{\partial T^{(0)}}{\partial x_i} + \left\langle C^{(1)} V_i^{(0)} \right\rangle \frac{\partial T^{(0)}}{\partial x_i} \right) \tag{A.19}$$

with

$$\lambda^*_{ij} = \frac{1}{|\Omega|} \int_{\Omega_\mathrm{w}+\Omega_\mathrm{s}} \lambda_{ij} \left(\delta_{kj} + \frac{\partial \chi_{\mathrm{II}\ k}}{\partial y_j} \right) \mathrm{d}\Omega.$$

Now, introducing the expression of $T^{(1)}$ in the convective term gives

$$\frac{\partial \left\langle C^{(0)} T^{(1)} \right\rangle_{\Omega_\mathrm{w}+\Omega_\mathrm{s}}}{\partial t} + \left(\left\langle C^{(0)} \right\rangle_{\Omega_\mathrm{a}} + \left\langle C^{(1)} \right\rangle_{\Omega_\mathrm{w}+\Omega_\mathrm{s}} \right) \frac{\partial T^{(0)}}{\partial t} \\ = \frac{\partial}{\partial x_i} \left(\lambda^{**\mathrm{eff}}_{ij} \frac{\partial T^{(0)}}{\partial x_j} \right) \\ - \left(\left\langle C^{(0)} V_i^{(0)} \right\rangle \frac{\partial \overline{T}^{(1)}}{\partial x_i} + \left\langle C^{(0)} V_i^{(1)} \right\rangle \frac{\partial T^{(0)}}{\partial x_i} + \left\langle C^{(1)} V_i^{(0)} \right\rangle \frac{\partial T^{(0)}}{\partial x_i} \right) \tag{A.20}$$

with

$$\lambda^{**\mathrm{eff}}_{ij} = \frac{1}{|\Omega|} \int_{\Omega_\mathrm{w}+\Omega_\mathrm{s}} \left[\lambda_{ij} \left(\mathrm{I}_{lj} + \frac{\partial \chi_{\mathrm{II}\ j}}{\partial y_l} \right) - C^{(0)} V_i^{(0)} \chi_{\mathrm{II}\ j} \right] \mathrm{d}\Omega.$$

Macroscopic models

By adding, member to member, Equation (A.8) to Equation (A.20) multiplied by ε, we get

$$\left\langle C^{(0)} \right\rangle_{\Omega_\mathrm{w}+\Omega_\mathrm{s}} \frac{\partial T^{(0)}}{\partial t} + \varepsilon \left[\frac{\partial \left\langle C^{(0)} T^{(1)} \right\rangle_{\Omega_\mathrm{w}+\Omega_\mathrm{s}}}{\partial t} + \left(\left\langle C^{(0)} \right\rangle_{\Omega_\mathrm{a}} + \left\langle C^{(1)} \right\rangle_{\Omega_\mathrm{w}+\Omega_\mathrm{s}} \right) \frac{\partial T^{(0)}}{\partial t} \right] \\ = - \left(\left\langle C^{(0)} V_i^{(0)} \right\rangle \frac{\partial T^{(0)}}{\partial x_i} \right) + \varepsilon \left[\frac{\partial}{\partial x_i} \left(\lambda^{\mathrm{eff}}_{ij} \frac{\partial T^{(0)}}{\partial x_j} \right) - \\ - \left(\left\langle C^{(0)} V_i^{(0)} \right\rangle \frac{\partial \overline{T}^{(1)}}{\partial x_i} + \left\langle C^{(0)} V_i^{(1)} \right\rangle \frac{\partial T^{(0)}}{\partial x_i} \right) + \left\langle C^{(1)} V_i^{(0)} \right\rangle \frac{\partial T^{(0)}}{\partial x_i} \right]. \tag{A.21}$$

Now, using the following relation

$$\left\langle C^{(0)}T^{(1)}\right\rangle_{\Omega_\mathrm{w}+\Omega_\mathrm{s}} = \left\langle C^{(0)}\right\rangle_{\Omega_\mathrm{w}+\Omega_\mathrm{s}} \overline{T}^{(1)} + \left\langle C^{(0)}\chi_{\mathrm{II}\ i}\frac{\partial T^{(0)}}{\partial x_i}\right\rangle_{\Omega_\mathrm{w}+\Omega_\mathrm{s}} \tag{A.22}$$

changes Equation (A.21) into

$$\begin{aligned}
&\left[\left\langle C^{(0)}\right\rangle_{\Omega_\mathrm{w}+\Omega_\mathrm{s}} + \varepsilon\left(\left\langle C^{(0)}\right\rangle_{\Omega_\mathrm{a}} + \left\langle C^{(1)}\right\rangle_{\mathrm{w,s}}\right)\right]\frac{\partial\left(\left\langle T^{(0)}+\varepsilon\overline{T}^{(1)}\right\rangle\right)}{\partial t} + \\
&\quad + \varepsilon\left\langle C^{(0)}\chi_{\mathrm{II}\ i}\right\rangle_{\Omega_\mathrm{w}+\Omega_\mathrm{s}}\frac{\partial^2 T^{(0)}}{\partial t\,\partial x_i} \\
&= -\left\langle C^{(0)}V_i^{(0)}\right\rangle\frac{\partial T^{(0)}}{\partial x_i} + \varepsilon\left[\frac{\partial}{\partial x_i}\left(\lambda_{ij}^{\mathrm{eff}}\frac{\partial T^{(0)}}{\partial x_j}\right) - \right. \\
&\quad \left. - \left(\left\langle C^{(0)}V_i^{(0)}\right\rangle\frac{\partial\overline{T}^{(1)}}{\partial x_i} + \varepsilon\left\langle C^{(1)}V_i^{(0)}\right\rangle\frac{\partial T^{(0)}}{\partial x_i}\right) + \left\langle C^{(0)}V_i^{(1)}\right\rangle\frac{\partial T^{(0)}}{\partial x_i}\right].
\end{aligned} \tag{A.23}$$

Finally, we obtain for the temperature field up to the second order ($T = T^{(0)} + \varepsilon T^{(1)}$)

$$\begin{aligned}
&\left(\langle C\rangle_{\Omega_\mathrm{w}+\Omega_\mathrm{s}} + \varepsilon\langle C\rangle_{\Omega_\mathrm{a}}\right)\frac{\partial T}{\partial t} + \varepsilon\,\langle C\chi_{\mathrm{II}\ i}\rangle_{\Omega_\mathrm{w}+\Omega_\mathrm{s}}\frac{\partial^2 T}{\partial t\,\partial x_i} \\
&\quad = \varepsilon\frac{\partial}{\partial x_i}\left(\lambda_{ij}^{**\mathrm{eff}}\frac{\partial T}{\partial x_j}\right) - \left(\langle CV_i\rangle_\Omega\frac{\partial T}{\partial x_i}\right) + \mathrm{O}(\varepsilon^2)
\end{aligned} \tag{A.24}$$

with

$$\lambda_{ij}^{**\mathrm{eff}} = \frac{1}{|\Omega|}\int_{\Omega_{\mathrm{w,s}}}\left[\lambda_{ij}\left(\mathrm{I}_{ij} + \frac{\partial\chi_{\mathrm{II}\ j}}{\partial y_l}\right) - C^{(0)}V_i^{(0)}\chi_{\mathrm{II}\ j}\right]\mathrm{d}\Omega.$$

The first term in the right-hand member of Equation (A-24) stands for the divergence of the macroscopic flux. This one appears as the volume averaging of the local flux. On an other hand, the macroscopic gradient of temperature of component $(\partial T^{(0)}/\partial x_i)$ is also the volume averaging of the local gradient $(\partial T^{(0)}/\partial x_i) + (\partial T^{(1)}/\partial y_i)$. Therefore the effective conductivity $\lambda^{**\mathrm{eff}}$ stands for the ratio between the volume-averaging heat flux and the volume-averaging temperature gradient.

References

Auriault, J.-L.: 1991a, Heterogeneous medium. Is an equivalent macroscopic description possible?, *Int. J. Engng. Sci.* **29**(7), 785–795.

Auriault, J.-L.: 1991b, Dynamic behaviour of porous media, in: J. Bear and P. Y. Corapcioglu (eds), *Transport Processes in Porous Media*, Kluwer Acad. Publ., Dordrecht, pp. 471–519.

Auriault, J.-L. and Boutin, C.: 1992, Deformable porous media with double porosity. Quasi-statics: I Coupling effects, *Transport in Porous Media* **7**, 63–82.

Auriault, J.-L. and Lewandowska, J.: 1995, Non-gaussian diffusion modeling in composite porous media by homogenization: tail effect, *Transport in Porous Media* **21**, 47–70.

Auriault, J.-L. and Lewandowska, J.: 1996, Diffusion/adsorption/advection in porous media: homogenization analysis, *Eur. J. Mec. A/Solids* **15**(4), 681–704.

Back, E. L.: 1988, Steam boxes in press sections-possibilities and limitations, *Appita* **41**(3), 217–223.

Bensoussan, A., Lions, J. L. and Papanicolaou, G.: 1978, *Asymptotic Analysis for Periodic Structures*, North-Holland, Amsterdam.

Bloch, J.-F.: 1995, Transferts de masse et de chaleur dans les milieux poreux déformables non saturés: application au pressage du papier, Thèse de Doctorat INP-G.

Campbell, W. B.: 1947, The physics of water removal, *Pulp and Paper Mag. Canada* **48**(3), 103–109.

Carlsson, G., Lindstrom, T. and Norman, B.: 1983, Some basics aspects on wet pressing of paper, *JPPS* **9**(4), 101.

Fekete, E.: 1975, Water removal on a grooved second press, Part II, in: *Int. Water Removal Symp. London*, Mar., p. 117.

Kataja, M., Hiltunen, K. and Timonen, J.: 1992, Flow of water and air in a compressible porous medium. A model of wet pressing of paper, *J. Phys. D: Appl. Phys.* **25**, 1053–1063.

Kaviany, M.: 1991, *Principles of Heat Transfer in Porous Media*, Mechanical Engineering Series, Springler-Verlag, New York.

Nilsson, P. and Larsson, K. O.: 1968, Paper web performance in a press nip, *Pulp and Paper Mag. Canada* **69**(24), 66–73.

Roux, J.-C.: 1986, Modélisation et optimisation du fonctionnement d'une section de presse(s) de machine à papier, Thèse de Doctorat de l'INP-Grenoble.

Wahlstrom, P. B.: 1960, A long term study of water removal and moisture distribution on a newsprint press section parts I and II, *Pulp and Paper Mag. Canada* **61**(8), 379–401, **61**(9), 418–451.

Wrist, P. E.: 1964, The present state of our knowledge of the fundamentals of wet pressing, *Pulp and Paper Mag. Canada* **65**(7), 284.